The Lambda Calculus

Your Favorite Language

Probably has lots of features:

- ~~Assignment (`x = x + 1`)~~
- ~~Booleans, integers, characters, strings, ...~~
- ~~Conditionals~~
- ~~Loops~~
- ~~`return`, `break`, `continue`~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~
- ...

1930s

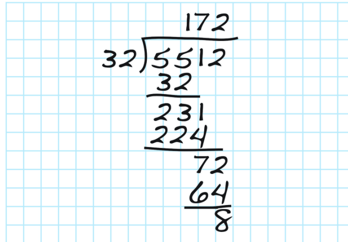
Which ones can we do without?

What is the **smallest universal language**?

What is computable?

Before 1930s

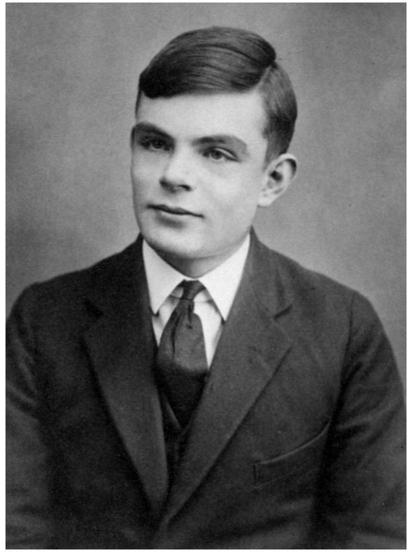
Informal notion of an **effectively calculable** function:



can be computed by a human with pen and paper, following an algorithm

1936: Formalization

What is the **smallest universal language**?



Alan Turing

The **Turing Machine** (https://en.wikipedia.org/wiki/Turing_machine)

Alonzo Church

The **Lambda Calculus**

The Next 700 Languages

Peter Landin

> *Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus.*

Peter Landin, 1966

The Lambda Calculus

Has one feature:

- Functions

"define" "call"

No, *really*

- ~~Assignment (`x = x + 1`)~~
- ~~Booleans, integers, characters, strings, ...~~
- ~~Conditionals~~
- ~~Loops~~
- ~~`return`, `break`, `continue`~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~
- ~~Reflection~~

More precisely, *only thing* you can do is:

- **Define** a function
- **Call** a function

Describing a Programming Language

- *Syntax:* what do programs look like?
- *Semantics:* what do programs mean?
  - *Operational semantics*: how do programs execute step-by-step?

Syntax: What Programs Look Like

EXPRESSION

```
e ::= x
    | \x -> e
    | (e1 e2)
```

$e := x$
$\mid$ function(x){ return e }
$\mid$ $e_1 (e_2)$

Programs are **expressions** `e` (also called **λ-terms**) of one of three kinds:

- **Variable**
  - `x`, `y`, `z`
- **Abstraction** (aka *nameless* function definition)
  - `\x -> e`
  - `x` is the *formal* parameter, `e` is the *body*
  - "for any `x` compute `e`"
- **Application** (aka function call)
  - `e1 e2`
  - `e1` is the *function*, `e2` is the *argument*
  - in your favorite language: `e1(e2)`

(Here each of `e`, `e1`, `e2` can itself be a variable, abstraction, or application)

Examples

```
\x -> x                -- The identity function
                       -- ("for any x compute x")

\x -> (\y -> y)        -- A function that returns the identity function

\f -> f (\x -> x)      -- A function that applies its argument
                       -- to the identity function
```

in

output

QUIZ

Which of the following terms are ~~syntactically incorrect~~?

NOT in λ-calc?

A. `\(\x -> x) -> y`

B. `\x -> x x`

C. `\x -> x (y x)`

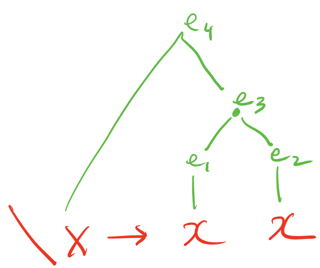
D. A and C

E. all of the above

e ::= x

| \x → e

| (e e)



Correct answer: **A**

Examples

```
\x -> x                -- The identity function
                       -- ("for any x compute x")

\x -> (\y -> y)        -- A function that returns the identity function

\f -> f (\x -> x)      -- A function that applies its argument
                       -- to the identity function
```

How do I define a function with two arguments?

- e.g. a function that takes `x` and `y` and returns `y`?

\x → (\y → y)

```
\x -> (\y -> y)      -- A function that returns the identity function
                     -- OR: a function that takes two arguments
                     -- and returns the second one!
```

$\boxed{((e\ \text{apple})\ \text{banana})} \leadsto \text{banana}$

How do I apply a function to two arguments?

- e.g. apply $\underbrace{\texttt{\textbackslash x -> (\textbackslash y -> y)}}_{e}$ to `apple` and `banana`?

```
(((\x -> (\y -> y)) apple) banana) -- first apply to apple,
                                   -- then apply the result to banana
```

Syntactic Sugar

instead of	we write
`\x -> (\y -> (\z -> e))`	`\x -> \y -> \z -> e`
`\x -> \y -> \z -> e`	`\x y z -> e`
`(((e1 e2) e3) e4)`	`e1 e2 e3 e4`

```
\x y -> y      -- A function that that takes two arguments
               -- and returns the second one...

(\x y -> y) apple banana -- ... applied to two arguments
```

Semantics : What Programs Mean

How do I "run" / "execute" a λ-term?

Think of middle-school algebra:

```
-- Simplify expression:

  (x + 2)*(3*x - 1)
 =
  ???
```

$$
\begin{aligned}
& (1+2) * (4-3) \\
=\ & 3 * (4-3) \\
=\ & 3 * 1 \\
=\ & 3
\end{aligned}
$$

$$
e_1 \to e_2 \to e_3 \to e_{final}
$$

Execute = rewrite step-by-step following simple rules, until no more rules apply "identity fun" \z → z

Rewrite Rules of Lambda Calculus

λ x → (λ x → x)

λ a → (λ b → b)

1. α-step (aka *renaming formals*)
2. β-step (aka *function call*)

But first we have to talk about **scope**

Semantics: Scope of a Variable

The part of a program where a **variable is visible**

In the expression `\x -> e`

binder

scope of x

- `x` is the newly introduced variable
- `e` is **the scope** of `x`
- any occurrence of `x` in `\x -> e` is **bound** (by the **binder** `\x`)

For example, `x` is bound in:

```
\x -> x
\x -> (\y -> x)
```

An occurrence of `x` in `e` is **free** if it's *not bound* by an enclosing abstraction

For example, `x` is free in:

```
x y                 -- no binders at all!
\y -> x y           -- no \x binder
(\x -> \y -> y) x   -- x is outside the scope of the \x binder;
                    -- intuition: it's not "the same" x
```

QUIZ

In the expression `(\x -> x) x`, is `x` *bound* or *free*?

1st 2nd

A. bound

B. free

C. first occurrence is bound, second is free

~~**D.** first occurrence is bound, second and third are free~~

~~**E.** first two occurrences are bound, third is free~~

Correct answer: **C**

Free Variables

An variable `x` is **free** in `e` if *there exists* a free occurrence of `x` in `e`

We can formally define the set of *all free variables* in a term like so:

```
FV(x)         = {x}
FV(\x -> e)   = FV(e) \ {x}
FV(e1 e2)     = FV(e1) + FV(e2)
```

Closed Expressions

If `e` has *no free variables* it is said to be **closed**

- Closed expressions are also called **combinators**

What is the shortest closed expression?

Answer: `\x -> x`

Rewrite Rules of Lambda Calculus

1. α-step (aka *renaming formals*)
2. β-step (aka *function call*)

Semantics: β-Reduction

```
(\x -> e1) e2   =b>   e1[x := e2]
```

where `e1[x := e2]` means “ `e1` with all *free* occurrences of `x` replaced with `e2` ”

Computation by *search-and-replace*:

- If you see an *abstraction* applied to an *argument*, take the *body* of the abstraction and replace all free occurrences of the *formal* by that *argument*
- We say that `(\x -> e1) e2` β-steps to `e1[x := e2]`

Examples

```
(\x -> x) apple
=b> apple
```

Is this right? Ask Elsa (http://goto.ucsd.edu:8095/index.html#?demo=blank.lc)!

```
(\f -> f (\x -> x)) (give apple)
=b> give apple (\x -> x)
```

QUIZ

```
(\x -> (\y -> y)) apple
=b> ???
```

A. `apple`

B. `\y -> apple`

C. `\x -> apple`

D. `\y -> y`

E. `\x -> y`

Correct answer: **D.**

QUIZ

```
(\x -> x (\x -> x)) apple
=b> ???
```

A. `apple (\x -> x)`

B. `apple (\apple -> apple)`

C. `apple (\x -> apple)`

D. `apple`

E. `\x -> x`

Correct answer: **A.**

A Tricky One

```
(\x -> (\y -> x)) y
=b> \y -> y
```

Is this right?

Something is Fishy

```
(\x -> (\y -> x)) y
=b> \y -> y
```

Is this right?

Problem: the *free* `y` in the argument has been **captured** by `\y`!

Solution: make sure that all *free variables* of the argument are different from the binders in the body.

Capture-Avoiding Substitution

We have to fix our definition of β-reduction:

```
(\x -> e1) e2   =b>   e1[x := e2]
```

where `e1[x := e2]` means ~~" `e1` with all *free* occurrences of `x` replaced with `e2` "~~

- `e1` with all *free* occurrences of `x` replaced with `e2` , **as long as** no free variables of `e2` get captured
- undefined otherwise

Formally:

```
x[x := e]             = e
y[x := e]             = y             -- assuming x /= y
(e1 e2)[x := e]       = (e1[x := e]) (e2[x := e])
(\x -> e1)[x := e]    = \x -> e1      -- why do we leave `e1` alone?
(\y -> e1)[x := e]
  | not (y in FV(e)) = \y -> e1[x := e]
  | otherise          = undefined     -- wait, but what do we do the
n???
```

Answer: We leave `e1` above alone even though it might contain `x`, because in `\x -> e1` every occurrence of `x` is bound by `\x` (hence, there are *no free occurrences* of `x`)

Rewrite Rules of Lambda Calculus

1. α-step (aka *renaming formals*)
2. β-step (aka *function call*)

Semantics: α-Renaming

```
\x -> e   =a>   \y -> e[x := y]
  where not (y in FV(e))
```

- We can rename a formal parameter and replace all its occurrences in the body
- We say that `\x -> e` α-steps to `\y -> e[x := y]`

Example:

```
\x -> x   =a>   \y -> y   =a>    \z -> z
```

All these expressions are ***α*-equivalent**

What's wrong with these?

```
-- (A)
\f -> f x    =a>   \x -> x x
```

Answer: it violates the side-condition for α-renaming that the new formal (`x`) must not occur freely in the body

```
-- (B)
(\x -> \y -> y) y   =a>   (\x -> \z -> z) z
```

Answer: we should only rename within the body of the abstraction; the second `y` is a free variable, and hence must remain unchanged

```
-- (C)
\x -> \y -> x y   =a>    \apple -> \orange -> apple orange
```

Answer: it's fine, but technically it's two α-steps and not one

The Tricky One

```
(\x -> (\y -> x)) y
=a> (\x -> (\z -> x)) y
=b> \z -> y
```

To avoid getting confused, you can always rename formals, so that different variables have different names!

Normal Forms

A **redex** is a λ-term of the form

```
(\x -> e1) e2
```

A λ-term is in **normal form** if it contains no redexes.

QUIZ

Which of the following term are **not** in *normal form* ?

A. `x`

B. `x y`

C. `(\x -> x) y`

D. `x (\y -> y)`

E. C and D

Answer: C

Semantics: Evaluation

A λ-term `e` **evaluates to** `e'` if

1. There is a sequence of steps

```
e =?> e_1 =?> ... =?> e_N =?> e'
```

where each `=?>` is either `=a>` or `=b>` and `N >= 0`

2. `e'` is in *normal form*

Examples of Evaluation

```
(\x -> x) apple
  =b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)
  =b> (\x -> x) (\x -> x)
  =b> \x -> x
```

```
(\x -> x x) (\x -> x)
  =b> (\x -> x) (\x -> x)
  =b> \x -> x
```

Elsa shortcuts

Named λ-terms:

```
let ID = \x -> x  -- abbreviation for \x -> x
```

To substitute name with its definition, use a `=d>` step:

```
ID apple
  =d> (\x -> x x) apple  -- expand definition
  =b> apple              -- beta-reduce
```

Evaluation:

- `e1 =*> e2` : `e1` reduces to `e2` in 0 or more steps
    - where each step is `=a>` , `=b>` , or `=d>`
- `e1 =~> e2` : `e1` evaluates to `e2`

What is the difference?

Non-Terminating Evaluation

```
(\x -> x x) (\x -> x x)
  =b> (\x -> x x) (\x -> x x)
```

Oops, we can write programs that loop back to themselves...

and never reduce to a normal form!

This combinator is called Ω

What if we pass Ω as an argument to another function?

```
let OMEGA = (\x -> x x) (\x -> x x)
```

```
(\x -> \y -> y) OMEGA
```

Does this reduce to a normal form? Try it at home!

Programming in λ-calculus

Real languages have lots of features

- Booleans
- Records (structs, tuples)
- Numbers
- **Functions** [we got those]
- Recursion

Lets see how to *encode* all of these features with the λ-calculus.

λ-calculus: Booleans

How can we encode Boolean values (`TRUE` and `FALSE`) as functions?

Well, what do we **do** with a Boolean `b` ?

Make a *binary choice*

- **`if`** `b` **`then`** `e1` **`else`** `e2`

Booleans: API

We need to define three functions

```
let TRUE  = ???
let FALSE = ???
let ITE   = \b x y -> ???  -- if b then x else y
```

such that

```
ITE TRUE apple banana =~> apple
ITE FALSE apple banana =~> banana
```

(Here, `let NAME = e` means `NAME` is an *abbreviation* for `e`)

Booleans: Implementation

```
let TRUE  = \x y -> x         -- Returns its first argument
let FALSE = \x y -> y         -- Returns its second argument
let ITE   = \b x y -> b x y   -- Applies condition to branches
                              -- (redundant, but improves readability)
```

Example: Branches step-by-step

```
eval ite_true:
  ITE TRUE e1 e2
  =d> (\b x y -> b    x  y) TRUE e1 e2    -- expand def ITE
  =b>   (\x y -> TRUE x  y)      e1 e2    -- beta-step
  =b>     (\y -> TRUE e1 y)         e2    -- beta-step
  =b>            TRUE e1 e2               -- expand def TRUE
  =d>     (\x y -> x) e1 e2               -- beta-step
  =b>       (\y -> e1)   e2               -- beta-step
  =b> e1
```

Example: Branches step-by-step

Now you try it!

Can you fill in the blanks to make it happen?
(http://goto.ucsd.edu:8095/index.html#?demo=ite.lc)

```
eval ite_false:
  ITE FALSE e1 e2
  =d> (\b x y -> b     x  y) FALSE e1 e2   -- expand def ITE
  =b>   (\x y -> FALSE x  y)       e1 e2   -- beta-step
  =b>     (\y -> FALSE e1 y)          e2   -- beta-step
  =b>            FALSE e1 e2               -- expand def FALSE
  =d>      (\x y -> y) e1 e2               -- beta-step
  =b>        (\y -> y)    e2               -- beta-step
  =b> e2
```

Boolean Operators

Now that we have `ITE` it's easy to define other Boolean operators:

```
let NOT = \b     -> ???

let AND = \b1 b2 -> ???

let OR  = \b1 b2 -> ???
```

```
let NOT = \b     -> ITE b FALSE TRUE

let AND = \b1 b2 -> ITE b1 b2 FALSE

let OR  = \b1 b2 -> ITE b1 TRUE b2
```

Or, since `ITE` is redundant:

```
let NOT = \b     -> b FALSE TRUE

let AND = \b1 b2 -> b1 b2 FALSE

let OR  = \b1 b2 -> b1 TRUE b2
```

Which definition to do you prefer and why?

Programming in λ-calculus

- **Booleans** [done]
- Records (structs, tuples)
- Numbers
- **Functions** [we got those]
- Recursion

λ-calculus: Records

Let's start with records with *two* fields (aka **pairs**)

What do we *do* with a pair?

1. **Pack two** items into a pair, then
2. **Get first** item, or
3. **Get second** item.

Pairs : API

We need to define three functions

```
let PAIR = \x y -> ???    -- Make a pair with elements x and y
                          -- { fst : x, snd : y }
let FST  = \p -> ???      -- Return first element
                          -- p.fst
let SND  = \p -> ???      -- Return second element
                          -- p.snd
```

such that

```
FST (PAIR apple banana) =~> apple
SND (PAIR apple banana) =~> banana
```

Pairs: Implementation

A pair of `x` and `y` is just something that lets you pick between `x` and `y`! (I.e. a function that takes a boolean and returns either `x` or `y`)

```
let PAIR = \x y -> (\b -> ITE b x y)
let FST  = \p -> p TRUE   -- call w/ TRUE, get first value
let SND  = \p -> p FALSE  -- call w/ FALSE, get second value
```

Exercise: Triples?

How can we implement a record that contains **three** values?

```
let TRIPLE = \x y z -> PAIR x (PAIR y z)
let FST3  = \t -> FST t
let SND3  = \t -> FST (SND t)
let TRD3  = \t -> SND (SND t)
```

Programming in λ-calculus

- **Booleans** [done]
- **Records** (structs, tuples) [done]
- Numbers
- **Functions** [we got those]
- Recursion

λ-calculus: Numbers

Let's start with **natural numbers** (0, 1, 2, …)

What do we *do* with natural numbers?

- Count: `0` , `inc`
- Arithmetic: `dec` , `+` , `-` , `*`
- Comparisons: `==` , `<=` , etc

Natural Numbers: API

We need to define:

- A family of **numerals**: `ZERO` , `ONE` , `TWO` , `THREE` , ...
- Arithmetic functions: `INC` , `DEC` , `ADD` , `SUB` , `MULT`
- Comparisons: `IS_ZERO` , `EQ`

Such that they respect all regular laws of arithmetic, e.g.

```
IS_ZERO ZERO       =~> TRUE
IS_ZERO (INC ZERO) =~> FALSE
INC ONE            =~> TWO
...
```

Natural Numbers: Implementation

Church numerals: *a number* `N` is encoded as a combinator that *calls a function on an argument* `N` *times*

```
let ONE   = \f x -> f x
let TWO   = \f x -> f (f x)
let THREE = \f x -> f (f (f x))
let FOUR  = \f x -> f (f (f (f x)))
let FIVE  = \f x -> f (f (f (f (f x))))
let SIX   = \f x -> f (f (f (f (f (f x)))))
...
```

QUIZ: Church Numerals

Which of these is a valid encoding of `ZERO` ?

- **A**: `let ZERO = \f x -> x`
- **B**: `let ZERO = \f x -> f`
- **C**: `let ZERO = \f x -> f x`
- **D**: `let ZERO = \x -> x`
- **E**: None of the above

Answer: A

Does this function look familiar?

Answer: It's the same as `FALSE` !

λ-calculus: Increment

```
-- Call `f` on `x` one more time than `n` does
let INC   = \n -> (\f x -> f (n f x))
```

Example:

```
eval inc_zero :
  INC ZERO
  =d> (\n f x -> f (n f x)) ZERO
  =b> \f x -> f (ZERO f x)
  =*> \f x -> f x
  =d> ONE
```

QUIZ

How shall we implement `ADD`?

A. `let ADD = \n m -> n INC m`

B. `let ADD = \n m -> INC n m`

C. `let ADD = \n m -> n m INC`

D. `let ADD = \n m -> n (m INC)`

E. `let ADD = \n m -> n (INC m)`

Answer: A

λ-calculus: Addition

```
--  Call `f` on `x` exactly `n + m` times
let ADD = \n m -> n INC m
```

Example:

```
eval add_one_zero :
  ADD ONE ZERO
  =~> ONE
```

QUIZ

How shall we implement `MULT`?

A. `let MULT = \n m -> n ADD m`

B. `let MULT = \n m -> n (ADD m) ZERO`

C. `let MULT = \n m -> m (ADD n) ZERO`

D. `let MULT = \n m -> n (ADD m ZERO)`

E. `let MULT = \n m -> (n ADD m) ZERO`

Answer: B or C

λ-calculus: Multiplication

```
--  Call `f` on `x` exactly `n * m` times
let MULT = \n m -> n (ADD m) ZERO
```

Example:

```
eval two_times_three :
  MULT TWO ONE
  =~> TWO
```

Programming in λ-calculus

- **Booleans** [done]
- **Records** (structs, tuples) [done]
- **Numbers** [done]
- **Functions** [we got those]
- Recursion

λ-calculus: Recursion

I want to write a function that sums up natural numbers up to `n`:

```
\n -> ...          -- 1 + 2 + ... + n
```

QUIZ

Is this a correct implementation of `SUM`?

```
let SUM = \n -> ITE (ISZ n)
            ZERO
            (ADD n (SUM (DEC n)))
```

A. Yes

B. No

No!

- Named terms in Elsa are just syntactic sugar
- To translate an Elsa term to λ-calculus: replace each name with its definition

```
\n -> ITE (ISZ n)
        ZERO
        (ADD n (SUM (DEC n))) -- But SUM is not a thing!
```

Recursion:

- Inside this function I want to call *the same function* on `DEC n`

Looks like we can't do recursion, because it requires being able to refer to functions *by name*, but in λ-calculus functions are *anonymous*.

Right?

λ-calculus: Recursion

Think again!

Recursion:

- ~~Inside this function I want to call *the same function* on `DEC n`~~
- Inside this function I want to call *a function* on `DEC n`
- *And BTW*, I want it to be the same function

Step 1: Pass in the function to call "recursively"

```
let STEP =
  \rec -> \n -> ITE (ISZ n)
                  ZERO
                  (ADD n (rec (DEC n))) -- Call some rec
```

Step 2: Do something clever to `STEP`, so that the function passed as `rec` itself becomes

```
\n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))
```

λ-calculus: Fixpoint Combinator

Wanted: a combinator `FIX` such that `FIX STEP` calls `STEP` with itself as the first argument:

```
FIX STEP
=*> STEP (FIX STEP)
```

(In math: a *fixpoint* of a function $f(x)$ is a point x, such that $f(x) = x$)

Once we have it, we can define:

```
let SUM = FIX STEP
```

Then by property of `FIX` we have:

```
SUM =*> STEP SUM -- (1)
```

```
eval sum_one:
  SUM ONE
  =*> STEP SUM ONE                   -- (1)
  =d> (\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ONE
  =b> (\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ONE
                                     -- ^^^ the magic happened!
  =b> ITE (ISZ ONE) ZERO (ADD ONE (SUM (DEC ONE)))
  =*> ADD ONE (SUM ZERO)             -- def of ISZ, ITE, DEC, ...
  =*> ADD ONE (STEP SUM ZERO)        -- (1)
  =d> ADD ONE
        ((\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ZERO)
  =b> ADD ONE ((\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ZERO)
  =b> ADD ONE (ITE (ISZ ZERO) ZERO (ADD ZERO (SUM (DEC ZERO))))
  =b> ADD ONE ZERO
  =~> ONE
```

How should we define `FIX` ???

The Y combinator

Remember Ω?

```
(\x -> x x) (\x -> x x)
=b> (\x -> x x) (\x -> x x)
```

This is *self-replcating code*! We need something like this but a bit more involved…

The Y combinator discovered by Haskell Curry:

```
let FIX   = \stp -> (\x -> stp (x x)) (\x -> stp (x x))
```

How does it work?

```
eval fix_step:
  FIX STEP
  =d> (\stp -> (\x -> stp (x x)) (\x -> stp (x x))) STEP
  =b> (\x -> STEP (x x)) (\x -> STEP (x x))
  =b> STEP ((\x -> STEP (x x)) (\x -> STEP (x x)))
  --       ^^^^^^^^^^ this is FIX STEP ^^^^^^^^^^^
```

That's all folks!
