Environments

Past three weeks

How to *use* essential language constructs?

- Data Types
- Recursion
- Higher-Order Functions

Next two weeks

How to *implement* language constructs?

- Local variables and scope
- Environments and Closures
- *(skip)* Type Inference

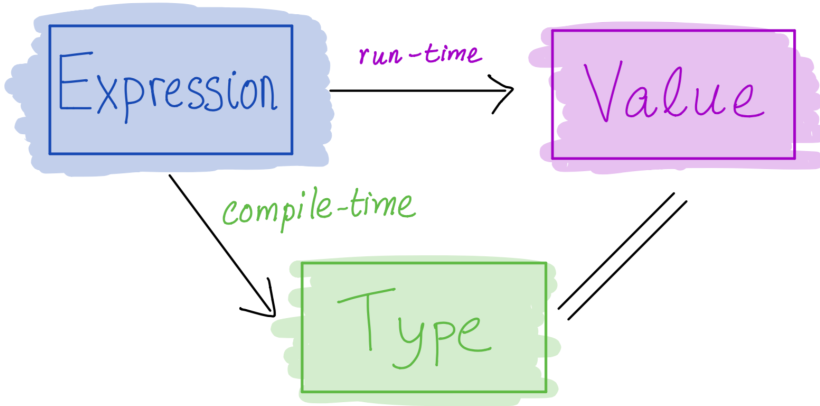
Interpreter

How do we *represent* and *evaluate* a program?

Roadmap: The Nano Language

Features of Nano:

1. **Arithmetic**
2. Variables
3. Let-bindings
4. Functions



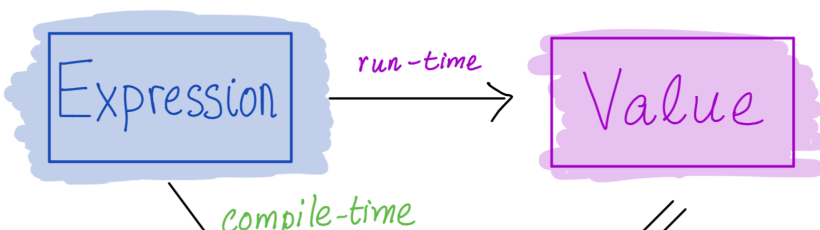
1. Nano: Arithmetic

A *grammar* of arithmetic expressions:

```
e ::= n
    | e1 + e2
    | e1 - e2
    | e1 * e2
```

Expressions		Values
`4`	`==>`	4
`4 + 12`	`==>`	16
`(4+12) - 5`	`==>`	11

Representing Arithmetic Expressions and Values



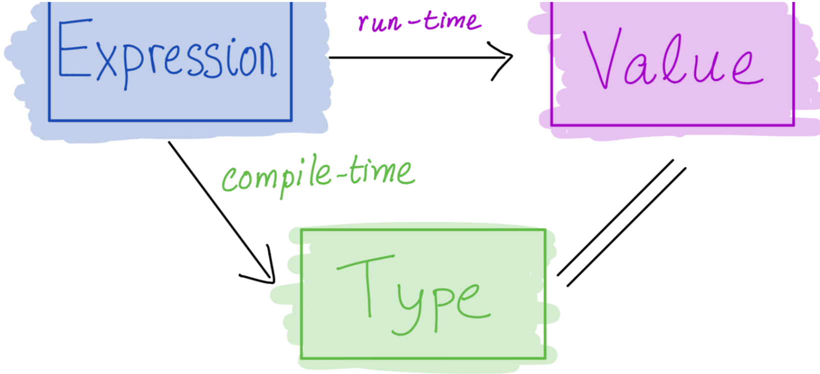
Lets *represent* arithmetic expressions as type

```
data Expr
  = ENum Int            -- ^ n
  | EAdd Expr Expr      -- ^ e1 + e2
  | ESub Expr Expr      -- ^ e1 - e2
  | EMul Expr Expr      -- ^ e1 * e2
```

Lets *represent* arithmetic values as a type

```
type Value = Int
```

Evaluating Arithmetic Expressions



We can now write a Haskell function to *evaluate* an expression:

```
eval :: Expr -> Value
eval (ENum n)      = n
eval (EAdd e1 e2) = eval e1 + eval e2
eval (ESub e1 e2) = eval e1 - eval e2
eval (EMul e1 e2) = eval e1 * eval e2
```

Alternative representation

Lets pull the *operators* into a separate type

```
data Binop = Add                          -- ^ `+`
           | Sub                          -- ^ `-`
           | Mul                          -- ^ `*`

data Expr  = ENum Int                      -- ^ n
           | EBin Binop Expr Expr          -- ^ e1 `op` e2
```

QUIZ

Evaluator for alternative representation

```
eval :: Expr -> Value
eval (ENum n)        = n
eval (EBin op e1 e2) = evalOp op (eval e1) (eval e2)
```

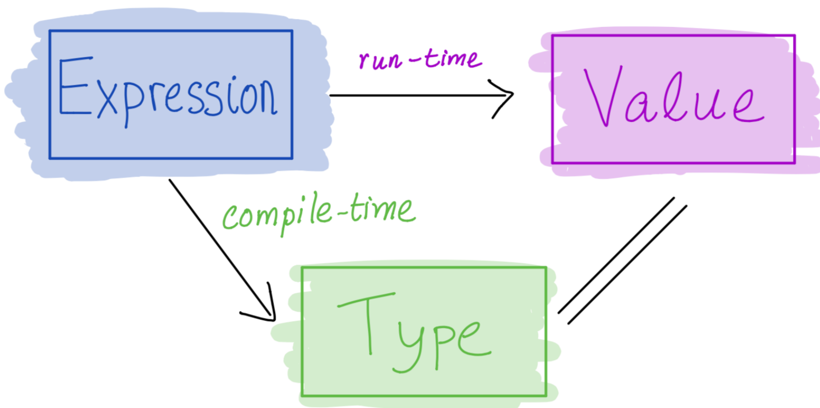
What is a suitable type for `evalOp` ?

```
{- 1 -} evalOp :: BinOp -> Value
{- 2 -} evalOp :: BinOp -> Value -> Value -> Value
{- 3 -} evalOp :: BinOp -> Expr  -> Expr -> Value
{- 4 -} evalOp :: BinOp -> Expr -> Expr -> Expr
{- 5 -} evalOp :: BinOp -> Expr -> Value
```

The Nano Language

Features of Nano:

1. Arithmetic *[done]*
2. **Variables**
3. Let-bindings
4. Functions
5. Recursion



2. Nano: Variables

Let's add variables and **`let`** bindings!

```
e ::= n                             -- OLD
    | e1 + e2
    | e1 - e2
    | e1 * e2
                                    -- NEW
    | x                             -- variables
```

Lets extend our datatype

```haskell
type Id = String

data Expr
  = ENum Int                -- OLD
  | EBin Binop Expr Expr
                            -- NEW
  | EVar Id                 -- variables
```

QUIZ

What should the following expression evaluate to?

```
x + 1
```

(A) `0`

(B) `1`

(C) Error

Environment

An expression is evaluated in an **environment**

- A **phone book** which maps *variables* to *values*

```
[ "x" := 0, "y" := 12, ...]
```

A type for *environments*

```
type Env = [(Id, Value)]
```

Evaluation in an Environment

We write

```
(eval env expr) ==> value
```

to mean

When `expr` is **evaluated in environment** `env` the result is `value`

So: when we have variables, we modify our evaluator (`eval`)

- to take an input environment `env` in which `expr` must be evaluated.

```
eval :: Env -> Expr -> Value
eval env expr = -- ... value-of-expr-in-env...
```

First, lets update the evaluator for the arithmetic cases `ENum` and `EBin`

```
eval :: Env -> Expr -> Value
eval env (ENum n)        = ???
eval env (EBin op e1 e2) = ???
```

QUIZ

What is a suitable `?value` such that

```
eval [ "x" := 0, "y" := 12, ...] (x + 1)  ==>  ?value
```

(A) `0`

(B) `1`

(C) Error

QUIZ

What is a suitable `env` such that

```
eval env (x + 1)   ==>   10
```

(A) `[]`

(B) `[x := 0, y := 9]`

(C) `[x := 9, y := 0]`

(D) `[x := 9, y := 10, z := 666]`

(E) `[y := 10, z := 666, x := 9]`

Evaluating Variables

Using the above intuition, lets update our evaluator to handle variables i.e. the `EVar` case:

```
eval env (EVar x)        = ???
```

Lets confirm that our `eval` is ok!

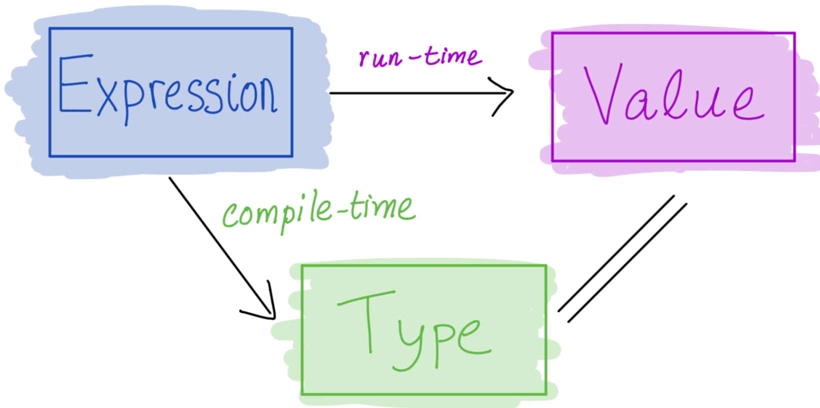
```
envA = []
envB = ["x" := 0 , "y" := 9]
envC = ["x" := 9 , "y" := 0]
envD = ["x" := 9 , "y" := 10 , "z" := 666]
envE = ["y" := 10, "z" := 666, "x" := 9  ]

-- >>> eval envA (EBin Add (EVar "x") (ENum 1))
-- >>> eval envB (EBin Add (EVar "x") (ENum 1))
-- >>> eval envC (EBin Add (EVar "x") (ENum 1))
-- >>> eval envD (EBin Add (EVar "x") (ENum 1))
-- >>> eval envE (EBin Add (EVar "x") (ENum 1))
```

The Nano Language

Features of Nano:

1. Arithmetic expressions *[done]*
2. Variables *[done]*
3. **Let-bindings**
4. Functions
5. Recursion



2. Nano: Variables

Let's add variables and **`let`** bindings!

```
e ::= n                              -- OLD
    | e1 + e2
    | e1 - e2
    | e1 * e2
    | x
                                     -- NEW
    | let x = e1 in e2
```

Lets extend our datatype

```haskell
type Id = String

data Expr
  = ENum Int                  -- OLD
  | EBin Binop Expr Expr
  | EVar Id
                             -- NEW
  | ELet Id Expr Expr
```

How should we extend `eval` ?

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  x + 1
```

(A) Error

(B) `1`

(C) `0`

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  let y = 100
  in
    x + y
```

(A) Error

(B) `0`

(C) `1`

(D) `100`

(E) `101`

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  let x = 100
  in
    x + 1
```

(A) Error

(B) `0`

(C) `1`

(D) `100`

(E) `101`

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  (let x = 100 in
   in
     x + 1
  )
  +
  x
```

(A) Error

(B) `1`

(C) `101`

(D) `102`

(E) `2`

Principle: Static/Lexical Scoping

Every variable *use* gets its value from a unique *definition*:

- "Nearest" `let` -binder in program *text*

Static means you can tell *without running the program*

Great for readability and debugging

1. Define *local* variables
2. Be sure *where* each variable got its value

Don't have to scratch head to figure where a variable got "assigned"

How to **implement** static scoping?

QUIZ

Lets re-evaluate the quizzes!

```
                    -- env
let x = 0
in                  -- ??? what env to use for `x + 1`?
  x + 1
```

(A) `env`

(B) `[ ]`

(C) `[ ("x" := 0) ]`

(D) `("x" := 0) : env`

(E) `env ++ ["x" := 0]`

QUIZ

```
                         -- env
let x = 0
in                         -- (x := 0) : env
  let y = 100
  in                         -- ??? what env to use for `x + y` ?
    x + y
```

(A) `("x" := 0) : env`

(B) `("y" := 100) : env`

(C) `("y" := 100) : ("x" := 0) : env`

(D) `("x" := 0) : ("y" := 100) : env`

(E) `[("y" := 100), ("x" := 0)]`

QUIZ

Lets re-evaluate the quizzes!

```
                  -- env
let x = 0
in                  -- ("x" := 0) : env
  let x = 100
  in                  -- ??? what env to use for `x + 1`?
    x + 1
```

(A) `("x" := 0) : env`

(B) `("x" := 100) : env`

(C) `("x" := 100) : ("x" := 0) : env`

(D) `("x" := 0) : ("x" := 100) : env`

(E) `[("x" := 100)]`

04-nano is out

Friday 3/4

Extending Environments

Lets fill in `eval` for the **`let`** `x = e1` **`in`** `e2` case!

let x = 10
in
x * x

```
eval env (ELet x e1 e2) = ???
```

1. **Evaluate** `e1` in `env` to get a value `v1`
2. **Extend** environment with value for `x` i.e. to `(x := v1) : env`
3. **Evaluate** `e2` using *extended* environment.

Lets make sure our tests pass!

Run-time Errors

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value
eval env (Num n)        = n
eval env (Var x)        = lookup x env     -- (A)
eval env (Bin op e1 e2) = evalOp op v1 v2  -- (B)
  where
    v1                  = eval env  e1     -- (C)
    v2                  = eval env  e2     -- (D)
eval env (Let x e1 e2)  = eval env1 e2
  where
    v1                  = eval env e1
    env1                = (x, v1) : env    --
```

QUIZ

Will `eval env expr` always return a `value` ? Or, can it *crash*?

(A) operation at `A` may fail **(B)** operation at `B` may fail **(C)** operation at `C` may fail **(D)** operation at `D` may fail **(E)** nah, its all good…, always returns a `Value`

Free vs bound variables

Undefined Variables

How do we make sure `lookup` doesn't cause a run-time error?

Bound Variables

Consider an expression `let x = e1 in e2`

- An occurrence of `x` is **bound** in `e2`
- i.e. when occurrence of form `let x = ... in ... x ...`
- i.e. when `x` occurs “under” a `let` binding for `x`.

Free Variables

An occurrence of `x` is **free** in `e` if it is **not bound** in `e`

Closed Expressions

An expression `e` is **closed** in environment `env`:

- If all **free** variables of `e` are defined in `env`

Successful Evaluation

`lookup` will never fail

- If `eval env e` is only called on `e` that is closed in `env`

QUIZ

Which variables occur free in the expression?

```
let y = (let x = 2
          in x      ) + x
in
  let x = 3
  in
   x + y
```

(A) None

(B) `x`

(C) `y`

(D) `x` and `y`

Exercise to try at home

Consider the function

isOK :: Expr → Bool

isOK e == TRUE only IF

e has NO free vars.

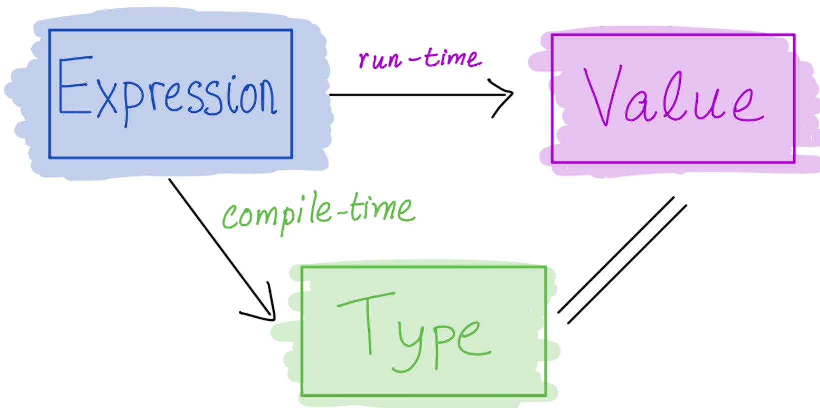
```
evaluate :: Expr -> Value
evaluate e
  | isOk e    = eval emptyEnv e
  | otherwise = error "Sorry! bad expression, it will crash `eval`!"
  where
    emptyEnv  = []                    -- has NO bindings
```

What should `isOk` check for? (Try to implement it for `nano` ...)

The Nano Language

Features of Nano:

1. Arithmetic expressions *[done]*
2. Variables *[done]*
3. Let-bindings *[done]*
4. **Functions**
5. Recursion



Nano: Functions

Let’s add

- *lambda abstraction* (aka function definitions)
- *application* (aka function calls)

\x -> e

$(e_1\ e_2)$

```
e ::= n                                -- OLD
    | e1 `op` e2
    | x
    | let x = e1 in e2
                                       -- NEW
    | \x -> e                          -- abstraction
    | e1 e2                            -- application
```

Example

```
let incr = \x -> x + 1
in
   incr 10
```

Representation

```
data Expr
  = ENum Int                   -- OLD
  | EBin Binop Expr Expr
  | EVar Id
  | ELet Id Expr Expr
                              -- NEW
  | ???                       -- abstraction \x -> e
  | ???                       -- application (e1 e2)
```

Representation

```haskell
data Expr
  = ENum Int                 -- OLD
  | EBin Binop Expr Expr
  | EVar Id
  | ELet Id Expr Expr
                            -- NEW
  | ELam Id Expr            -- abstraction \x -> e
  | EApp Expr Expr          -- application (e1 e2)
```

Example

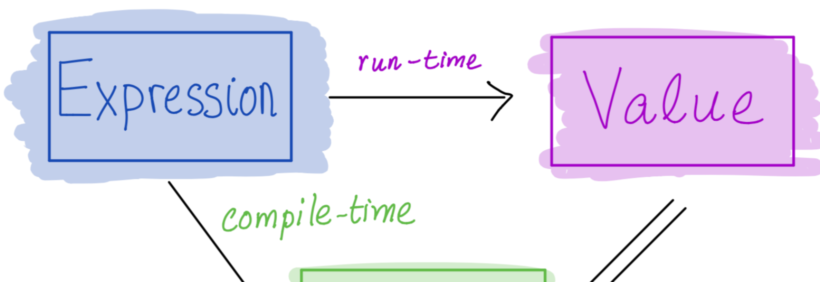
```haskell
let incr = \x -> x + 1
in
   incr 10
```

is represented as

```haskell
ELet "incr" (ELam "x" (EBin Add (EVar "x") (ENum 1)))
  (
    EApp (EVar "incr") (ENum 10)
  )
```

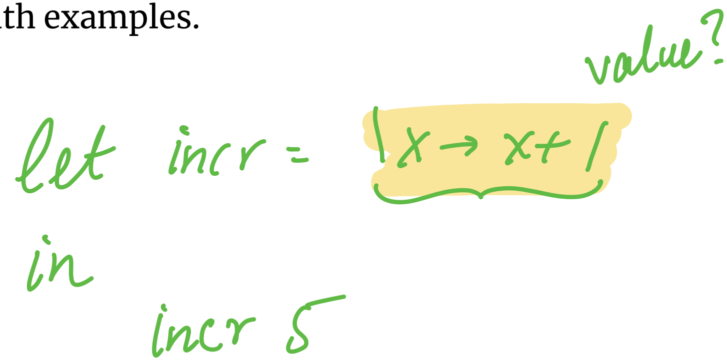
Functions are Values

Recall the trinity



But... what is the *value* of a function?

Lets build some intuition with examples.



QUIZ

What does the following expression evaluate to?

```
let incr = \x -> x + 1      -- abstraction ("definition")
in
   incr 10                  -- application ("call")
```

(incr := ??? : ENV)

(A) Error/Undefined

(B) `10`

(C) `11`

(D) `0`

(E) `1`

What is the Value of `incr`?

- Is it an `Int` ?
- Is it a `Bool` ?
- Is it a ???

What information do we need to store (in the `Env`) about `incr` ?

incr is a FUNCTION & what it does

"x" → x+1

\x

A Function's Value is its Code

```
                                 -- env
let incr = \x -> x + 1
in                               -- ("incr" := <code>) : env
    incr 10                      -- evaluate <code> with parameter := 10
```

What information do we store about `<code>` ?

A Call's Value

How to evaluate the "call" `incr 10` ?

1. Lookup the `<code>` i.e. `<param, body>` for `incr` (stored in the environment),

2. Evaluate `body` with `param` set to `10`!

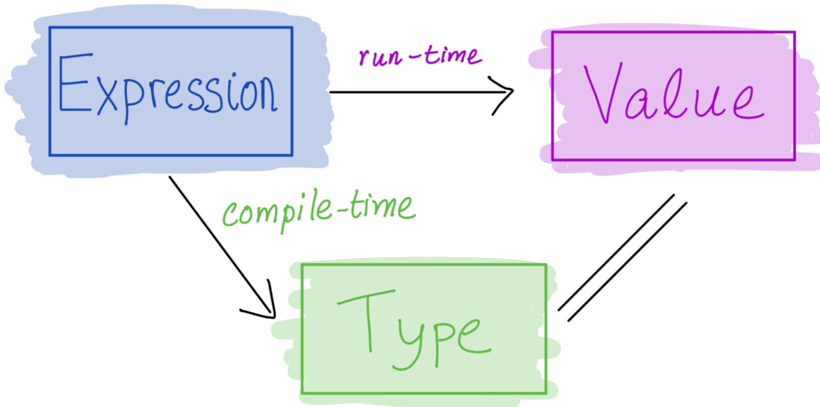
Two kinds of Values

We now have *two* kinds of Values

```
v ::= n              -- OLD
    | <x, e>         -- <param, body>
```

1. Plain `Int` (as before)
2. A function's "code": a pair of "parameter" and "body-expression"

```
data Value
  = VInt  Int               -- OLD
  | VCode Id Expr           -- <x, e>
```



Evaluating Lambdas and Applications

```haskell
eval :: Env -> Expr -> Value
                                            -- OLD
eval env (ENum n)         = ???
eval env (EVar x)         = ???
eval env (EBin op e1 e2) = ???
eval env (ELet x  e1 e2) = ???
                                            -- NEW
eval env (ELam x e)       = ???
eval env (EApp e1 e2)     = ???
```

Lets make sure our tests work properly!

```haskell
exLam1 = ELet "incr" (ELam "x" (EBin Add (EVar "x") (ENum 1)))
           (
             EApp (EVar "incr") (ENum 10)
           )

-- >>> eval [] exLam1
-- 11
```

QUIZ

What should the following evaluate to?

```
let c = 1
in
   let inc = \x -> x + c
   in
      inc 10
```

(A) Error/Undefined

(B) `10`

(C) `11`

(D) `0`

(E) `1`

```
exLam2 = ELet "c" (ENum 1)
           (ELet "incr" (ELam "x" (EBin Add (EVar "x") (EVar "c")))
              (
              EApp (EVar "incr") (ENum 10)
              )
           )

-- >>> eval [] exLam2
-- ???
```

QUIZ

And what should *this* expression evaluate to?

```
let c = 1
in
   let inc = \x -> x + c
   in
      let c = 100
      in
         inc 10
```

(A) Error/Undefined

(B) `110`

(C) `11`

The "Immutability Principle"

A function's behavior should *never change*

- A function must *always* return the same output for a given input

Why?

```
> myFunc 10
0
```

```
> myFunc 10
10
```

Oh no! How to find the bug? Is it

- In `myFunc` or
- In a global variable or
- In a library somewhere else or
- ...

My worst debugging nightmare

Colbert "Immutability Principle" (https://youtu.be/CWqzLgDc030?t=628)

The Immutability Principle ?

How does our `eval` work?

```
exLam3 = ELet "c" (ENum 1)
           (
           ELet "incr" (ELam "x" (EBin Add (EVar "x") (EVar "c")))
              (
               ELet "c" (ENum 100)
                 (
                 EApp (EVar "incr") (ENum 10)
                 )
              )
           )

-- >>> eval [] exLam3
-- ???
```

Oops?

```
                                  -- []
let c = 1
in                                -- ["c" := 1]
   let inc = \x -> x + c
   in                             -- ["inc" := <x, x+c>, c := 1]
      let c = 100
      in                          -- ["c" := 100, "inc" := <x, x+c", "c" := 1]
<<< env
        inc 10
```

And so we get

```
eval env (inc 10)

  ==> eval ("x" := 10 : env) (x + c)

  ==> 10 + 100

  ==> 110
```

Ouch.

Enforcing Immutability with Closures

How to enforce immutability principle

- `inc 10` **always** returns `11`?

Key Idea: Closures

At definition: *Freeze* the environment the function's value

At call: Use the *frozen* environment to evaluate the *body*

Ensures that `inc 10` *always* evaluates to the *same* result!

```
                              -- []
let c = 1
in                            -- ["c" := 1]
  let inc = \x -> x + c
  in                          -- ["inc" := <frozenv, x, x+c>, c := 1]  <<< froz
env = ["c" := 1]
    let c = 100
    in                        -- ["c" := 100, "inc" := <frozenv, x, x+c>, "c" :
= 1]
      inc 10
```

Now we evaluate

```
eval env (inc 10)

  ==> eval ("x" := 10 : frozenv) (x + c)  where frozenv = ["c" := 1]

  ==> 10 + 1

  ==> 1
```

tada!

Representing Closures

Lets change the `Value` datatype to also store an `Env`

```
data Value
  = VInt  Int           -- OLD
  | VClos Env Id Expr   -- <frozenv, param, body>
```

Evaluating Function Definitions

How should we fix the definition of `eval` for `ELam` ?

```
eval :: Env -> Expr -> Value

eval env (ELam x e) = ???
```

Hint: What value should we *bind* `incr` to in our example above?

(Recall **At definition** *freeze* the environment the function's value)

Evaluating Function Calls

How should we fix the definition of `eval` for `EApp` ?

```
eval :: Env -> Expr -> Value

eval env (EApp e1 e2) = ???
```

(Recall **At call:** Use the *frozen* environment to evaluate the *body*)

Hint: What value should we *evaluate* `incr 10` to?

1. Evaluate `incr` to get `<frozenv, "x", x + c>`
2. Evaluate `10` to get `10`
3. Evaluate `x + c` in `x:=10 : frozenv`

Let's generalize that recipe!

1. Evaluate `e1` to get `<frozenv, param, body>`
2. Evaluate `e2` to get `v2`
3. Evaluate `body` in `param := v2 : frozenv`

Immutability Achieved

Lets put our code to the test!

```
exLam3 = ELet "c" (ENum 1)
           (
           ELet "incr" (ELam "x" (EBin Add (EVar "x") (EVar "c")))
              (
               ELet "c" (ENum 100)
                 (
                 EApp (EVar "incr") (ENum 10)
                 )
              )
           )

-- >>> eval [] exLam3
-- ???
```

QUIZ

What should the following evaluate to?

```
let add = \x -> (\y -> x + y)
in
  let add10 = add 10
  in
    let add20 = add 20
    in
      (add10 100) + (add20 1000)
```

A. `1100`

B. `1110`

C. `1120`

D. `1130`

E. `1140`

Functions Returning Functions Achieved!

```
exLam4 = ...

-- >>> eval [] exLam4
```

Practice

What should the following evaluate to?

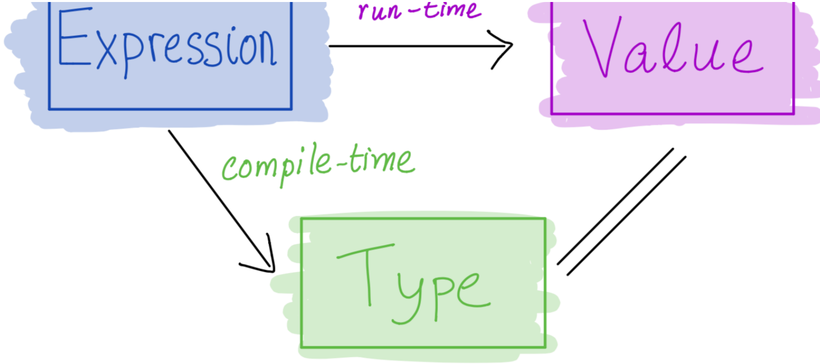
```
let add = \x -> (\y -> x + y)
in
  let add10 = add 10
  in
    let doTwice = \f -> (\x -> f (f x))
    in
      doTwice add10 100
```

Functions Accepting Functions Achieved!

```
exLam5 = ...
```

```
-- >>> eval [] exLam4
```

The Nano Language

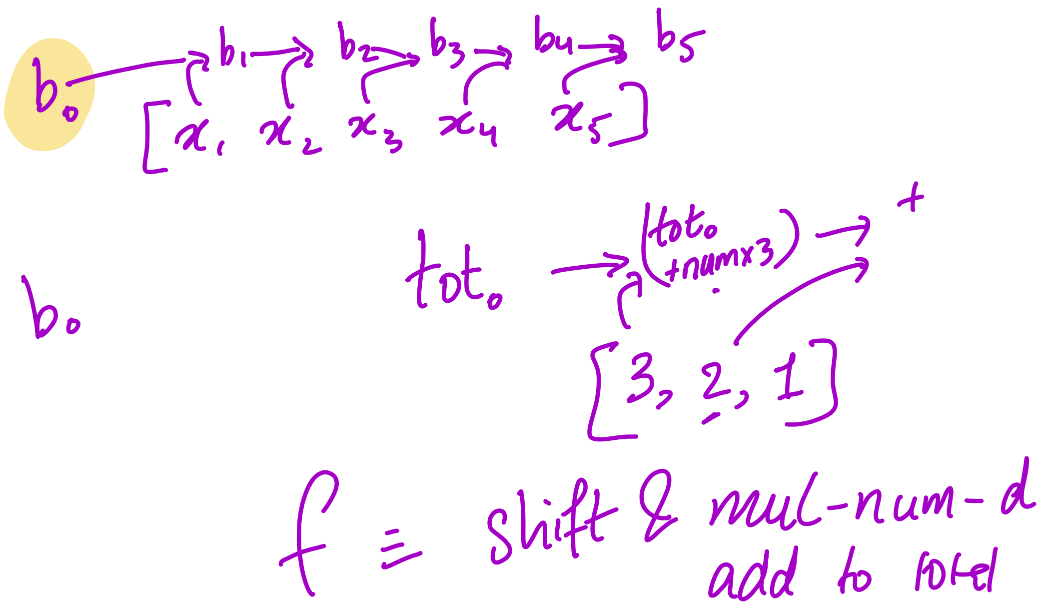


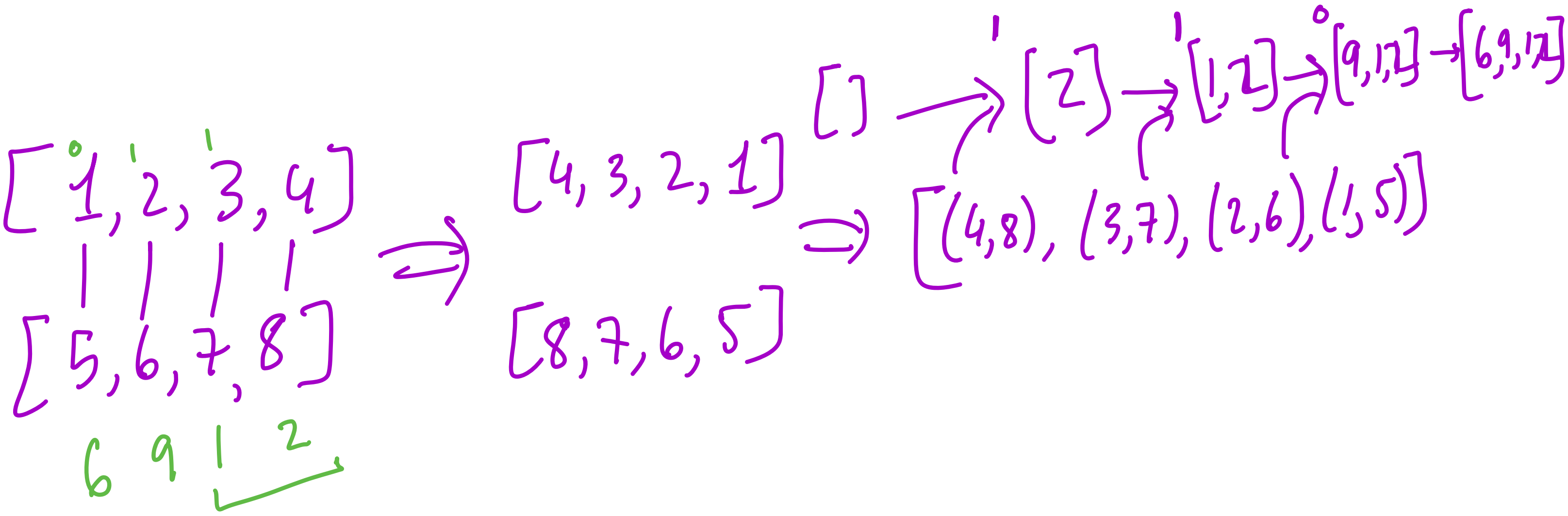
Features of Nano:

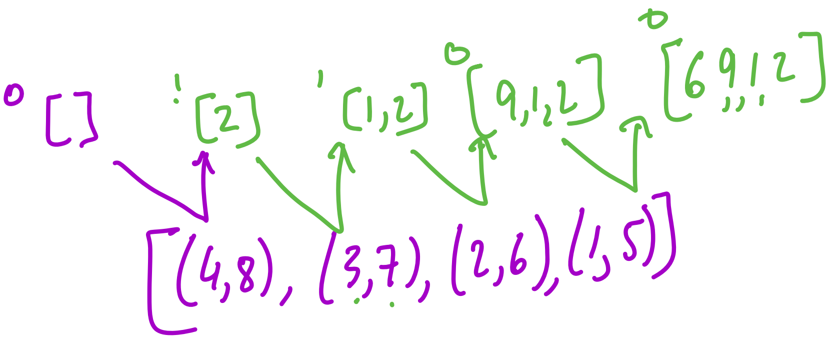
1. Arithmetic expressions *[done]*
2. Variables *[done]*
3. Let-bindings *[done]*
4. Functions *[done]*
5. **Recursion**

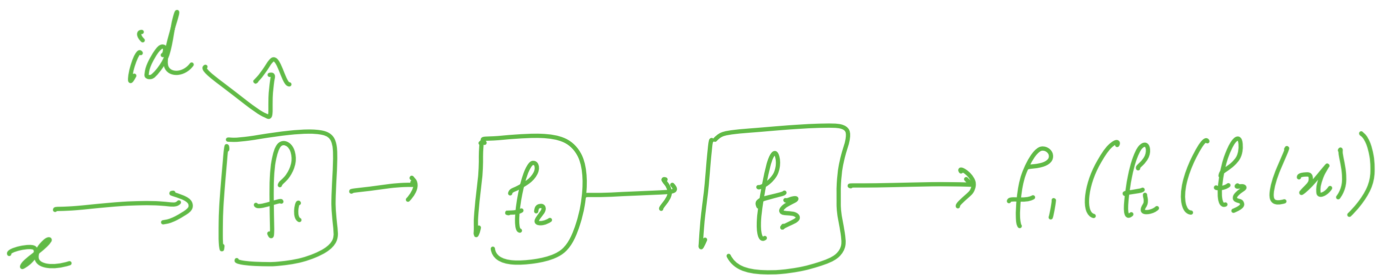
… You figure it out **Hw4** … :-)

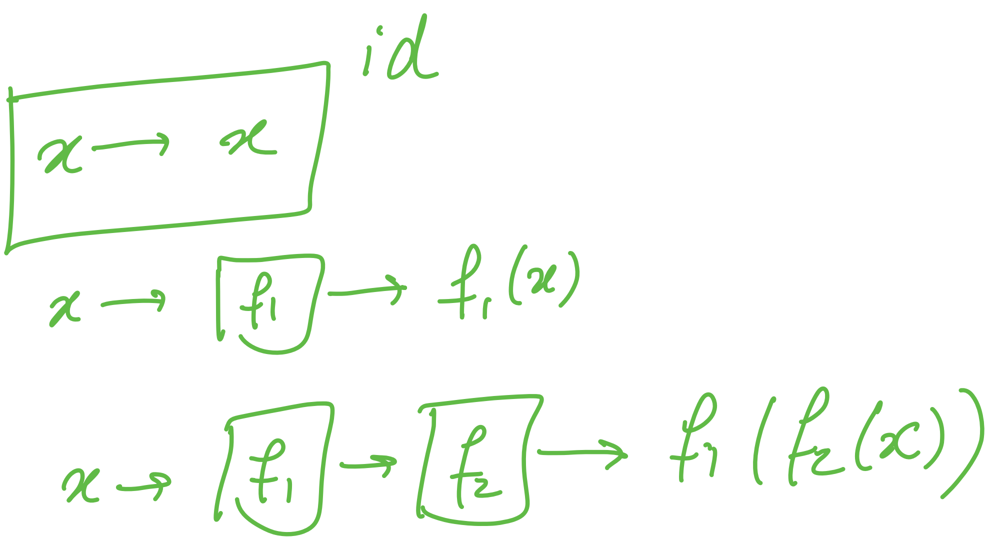
(https://ucsd-cse130.github.io/wi22/feed.xml) (https://twitter.com/ranjitjhala) (https://plus.google.com/u/0/104385825850161331469) (https://github.com/ranjitjhala)




[1, 2, 3, 4]
[5, 6, 7, 8]
[4, 3, 2, 1]
[8, 7, 6, 5]
[(4,8), (3,7), (2,6), (1,5)]
[] → [2] → [1, 2] → [9, 1, 2] → [6, 9, 1, 2]
6 9 1 2


[] [2] [1, 2] [9, 1, 2] [6, 9, 1, 2]
[(4,8), (3,7), (2,6), (1,5)]


id
x
f1
f2
f3
f1 (f2 (f3 (x)))


id
x → x
x → f1 → f1 (x)
x → f1 → f2 → f1 (f2 (x))

$$b_0 \equiv \backslash x_0 \to x_0$$

$$b_1 \equiv op\ b_0\ f_1 \equiv \backslash x_1 \to f_1(x_1)$$

$$b_2 \equiv op\ b_1\ f_2 \equiv \backslash x_2 \to f_1(f_2(x_2))$$

$$b_3 \equiv op\ b_2\ f_3 \equiv \backslash x_3 \to f_1(f_2(f_3(x_3)))$$

$$b_4 \equiv op\ b_3\ f_4 \equiv \backslash x_4 \to f_1(f_2(f_3(f_4\ x_4))))$$

$$\backslash x \to b_3(f_4\ x)$$