Lambda Calculus

Your Favorite Language

Probably has lots of features:

- ~~Assignment (`x = x + 1`)~~
- ~~Booleans, integers, characters, strings, …~~
- ~~Conditionals~~
- ~~Loops~~
- ~~`return`, `break`, `continue`~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~
- …

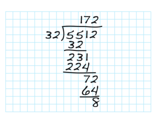
Which ones can we do without?

What is the **smallest universal language**?

What is computable?

Before 1930s

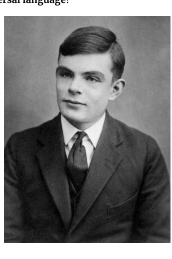
Informal notion of an **effectively calculable** function:



can be computed by a human with pen and paper, following an algorithm

1936: Formalization

What is the **smallest universal language**?



Alan Turing

Alonzo Church

The Next 700 Languages

Peter Landin

> *Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus.*

Peter Landin, 1966

The Lambda Calculus

Has one feature:

- Functions

No, *really*

- ~~Assignment (`x = x + 1`)~~
- ~~Booleans, integers, characters, strings, …~~
- ~~Conditionals~~
- ~~Loops~~
- ~~`return`, `break`, `continue`~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~
- ~~Reflection~~

More precisely, *only thing* you can do is:

- **Define** a function
- **Call** a function

Describing a Programming Language

- *Syntax:* what do programs look like?
- *Semantics:* what do programs mean?
  - *Operational semantics*: how do programs execute step-by-step?

Syntax: What Programs Look Like

```
e ::= x           -- variable 'x'
    | (\x -> e)   -- function that takes a parameter 'x' and returns 'e'
    | (e1 e2)     -- call (function) 'e1' with argument 'e2'
```

Programs are **expressions** `e` (also called **λ-terms**) of one of three kinds:

- **Variable**
  - `x`, `y`, `z`
- **Abstraction** (aka *nameless* function definition)
  - `(\x -> e)`
  - `x` is the *formal* parameter, `e` is the *body*
  - "for any `x` compute `e`"
- **Application** (aka function call)
  - `(e1 e2)`
  - `e1` is the *function*, `e2` is the *argument*
  - in your favorite language: `e1(e2)`

(Here each of `e`, `e1`, `e2` can itself be a variable, abstraction, or application)

Examples

```
(\x -> x)             -- The identity function (id) that returns its input

(\x -> (\y -> y))     -- A function that returns (id)

(\f -> (f (\x -> x))) -- A function that applies its argument to id
```

QUIZ

Which of the following terms are syntactically **incorrect**?

A. `(\(\x -> x) -> y)`

B. `(\x -> (x x))`

C. `(\x -> (x (y x)))`

D. A and C

E. all of the above

Examples

```
(\x -> x)             -- The identity function (id) that returns its input

(\x -> (\y -> y))     -- A function that returns (id)

(\f -> (f (\x -> x))) -- A function that applies its argument to id
```

How do I define a function with two arguments?

- e.g. a function that takes `x` and `y` and returns `y`?

```
(\x -> (\y -> y))   -- A function that returns the identity function
                    -- OR: a function that takes two arguments
                    -- and returns the second one!
```

How do I apply a function to two arguments?

- e.g. apply `(\x -> (\y -> y))` to `apple` and `banana`?

```
(((\x -> (\y -> y)) apple) banana) -- first apply to apple,
                                   -- then apply the result to banana
```

Syntactic Sugar

instead of	we write
`\x -> (\y -> (\z -> e))`	`\x -> \y -> \z -> e`
`\x -> \y -> \z -> e`	`\x y z -> e`
`(((e1 e2) e3) e4)`	`e1 e2 e3 e4`

```
\x y -> y     -- A function that that takes two arguments
              -- and returns the second one...

(\x y -> y) apple banana -- ... applied to two arguments
```

Semantics : What Programs Mean

How do I "run" / "execute" a λ-term?

Think of middle-school algebra:

```
   (1 + 2) * ((3 * 8) - 2)
==
   3       * ((3 * 8) - 2)
==
   3       * ( 24     - 2)
==
   3       * 22
==
   66
```

Execute = rewrite step-by-step

- Following simple *rules*
- until no more rules *apply*

Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

But first we have to talk about **scope**

Semantics: Scope of a Variable

The part of a program where a **variable is visible**

In the expression `(\x -> e)`

- `x` is the newly introduced variable
- `e` is **the scope** of `x`
- any occurrence of `x` in `(\x -> e)` is **bound** (by the **binder** `\x`)

For example, `x is bound` in:

```
(\x -> x)

(\x -> (\y -> x))
```

An occurrence of `x` in `e` is **free** if it's *not bound* by an enclosing abstraction

For example, `x` is free in:

```
(x y)                -- no binders at all!

(\y -> (x y))        -- no \x binder

((\x -> (\y -> y)) x) -- x is outside the scope of the \x binder;
                      -- intuition: it's not "the same" x
```

QUIZ

Is `x` *bound* or *free* in the expression `((\x -> x) x)`?

A. first occurrence is bound, second is bound

B. first occurrence is bound, second is free

C. first occurrence is free, second is bound

D. first occurrence is free, second is free

EXERCISE: Free Variables

An variable `x` is **free** in `e` if *there exists* a free occurrence of `x` in `e`

We can formally define the set of *all free variables* in a term like so:

```
FV(x)       = ???
FV(\x -> e) = ???
FV(e1 e2)   = ???
```

Closed Expressions

If `e` has *no free variables* it is said to be **closed**

- Closed expressions are also called **combinators**

What is the shortest closed expression?

Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

Semantics: Redex

A **redex** is a term of the form

```
((\x -> e1) e2)
```

A *function* `(\x -> e1)`

- `x` is the *parameter*
- `e1` is the *returned* expression

Applied to an argument `e2`

- `e2` is the *argument*

Semantics: β-Reduction

A **redex** b-steps to another term …

```
(\x -> e1) e2   =b>   e1[x := e2]
```

where `e1[x := e2]` means

"`e1` with all *free* occurrences of `x` replaced with `e2`"

Computation by *search-and-replace*:

If you see an *abstraction* applied to an *argument*,

- In the *body of the abstraction*
- Replace all *free occurrences* of the *formal* by that *argument*

We say that `(\x -> e1) e2` β-steps to `e1[x := e2]`

Redex Examples

```
((\x -> x) apple)

=b> apple
```

Is this right? Ask Elsa

QUIZ

```
((\x -> (\y -> y)) apple)

=b> ???
```

A. `apple`

B. `\y -> apple`

C. `\x -> apple`

D. `\y -> y`

E. `\x -> y`

QUIZ

```
(\x -> (((y x) y) x)) apple
=b> ???
```

A. `(((apple apple) apple) apple)`

B. `(((y apple) y) apple)`

C. `(((y y) y) y)`

D. `apple`

QUIZ

```
((\x -> (x (\x -> x))) apple)

=b> ???
```

A. `(apple (\x -> x))`

B. `(apple (\apple -> apple))`

C. `(apple (\x -> apple))`

D. `apple`

E. `(\x -> x)`

EXERCISE

What is a λ-term `fill_this_in` such that

```
fill_this_in apple
=b> banana
```

ELSA: https://goto.ucsd.edu/elsa/index.html

Click here to try this exercise

A Tricky One

```
((\x -> (\y -> x)) y)

=b> \y -> y
```

Is this right?

Something is Fishy

```
(\x -> (\y -> x)) y

=b> (\y -> y)
```

Is this right?

Problem: The *free* `y` in the argument has been **captured** by `\y` in *body*!

Solution: Ensure that *formals* in the body are **different from** *free-variables* of argument!

Capture-Avoiding Substitution

We have to fix our definition of β-reduction:

```
(\x -> e1) e2   =b>   e1[x := e2]
```

where `e1[x := e2]` means "~~`e1` with all *free* occurrences of `x` replaced with `e2`~~"

- `e1` with all *free* occurrences of `x` replaced with `e2`
- **as long as** no free variables of `e2` get captured

Formally:

```
x[x := e]            = e

y[x := e]            = y                -- as x /= y

(e1 e2)[x := e]      = (e1[x := e]) (e2[x := e])

(\x -> e1)[x := e]   = (\x -> e1)       -- Q: Why leave `e1` unchanged?

(\y -> e1)[x := e]
  | not (y in FV(e)) = \y -> e1[x := e]
```

Oops, but what to do if `y` is in the *free-variables* of `e`?

- i.e. if `\y -> ...` may *capture* those free variables?

Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

Semantics: α-Renaming

```
\x -> e   =a>   \y -> e[x := y]
  where not (y in FV(e))
```

- We rename a formal parameter `x` to `y`
- By replace all occurrences of `x` in the body with `y`
- We say that `\x -> e` α-steps to `\y -> e[x := y]`

Example:

```
(\x -> x)   =a>   (\y -> y)   =a>   (\z -> z)
```

All these expressions are α-**equivalent**

What's wrong with these?

```
-- (A)
(\f -> (f x))    =a>   (\x -> (x x))

-- (B)
((\x -> (\y -> y)) y)   =a>   ((\x -> (\z -> z)) z)
```

Tricky Example Revisited

```
((\x -> (\y -> x)) y)
                          -- rename 'y' to 'z' to avoid capture
=a> ((\x -> (\z -> x)) y)
                          -- now do b-step without capture!
=b> (\z -> y)
```

To avoid getting confused,

- you can **always rename** formals,
- so different **variables** have different **names**!

Normal Forms

Recall **redex** is a λ-term of the form

```
((\x -> e1) e2)
```

A λ-term is in **normal form** if it *contains no redexes*.

QUIZ

Which of the following term are **not** in *normal form* ?

A. `x`

B. `(x y)`

C. `((\x -> x) y)`

D. `(x (\y -> y))`

E. C and D

Semantics: Evaluation

A λ-term `e` **evaluates to** `e'` if

1. There is a sequence of steps

```
e =?> e_1 =?> ... =?> e_N =?> e'
```

where each `=?>` is either `=a>` or `=b>` and `N >= 0`

2. `e'` is in *normal form*

Examples of Evaluation

```
((\x -> x) apple)

  =b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)

  =?> ???
```

```
(\x -> x x) (\x -> x)
  =?> ???
```

Elsa shortcuts

Named λ-terms:

```
let ID = (\x -> x)   -- abbreviation for (\x -> x)
```

To substitute name with its definition, use a `=d>` step:

```
(ID apple)
  =d> ((\x -> x) apple)   -- expand definition
  =b> apple               -- beta-reduce
```

Evaluation:

- `e1 =*> e2` : `e1` reduces to `e2` in 0 or more steps
  - where each step is `=a>` , `=b>` , or `=d>`
- `e1 =~> e2` : `e1` evaluates to `e2` and `e2` is **in normal form**

EXERCISE

Fill in the definitions of `FIRST` , `SECOND` and `THIRD` such that you get the following behavior in `elsa`

```
let FIRST  = fill_this_in
let SECOND = fill_this_in
let THIRD  = fill_this_in

eval ex1 :
  FIRST apple banana orange
  =*> apple

eval ex2 :
  SECOND apple banana orange
  =*> banana

eval ex3 :
  THIRD apple banana orange
  =*> orange
```

ELSA: https://goto.ucsd.edu/elsa/index.html

Click here to try this exercise

Non-Terminating Evaluation

```
((\x -> (x x)) (\x -> (x x)))

  =b> ((\x -> (x x)) (\x -> (x x)))
```

Some programs loop back to themselves ... *never* reduce to a normal form!

This combinator is called Ω

What if we pass Ω as an argument to another function?

```
let OMEGA = ((\x -> (x x)) (\x -> (x x)))
```

```
((\x -> (\y -> y)) OMEGA)
```

Does this reduce to a normal form? Try it at home!

Programming in λ-calculus

Real languages have lots of features

- Booleans
- Records (structs, tuples)
- Numbers
- Lists
- **Functions** [we got those]
- Recursion

Lets see how to *encode* all of these features with the λ-calculus.

Syntactic Sugar

instead of	we write
`\x -> (\y -> (\z -> e))`	`\x -> \y -> \z -> e`
`\x -> \y -> \z -> e`	`\x y z -> e`
`(((e1 e2) e3) e4)`	`e1 e2 e3 e4`

```
\x y -> y      -- A function that that takes two arguments
               -- and returns the second one...

(\x y -> y) apple banana -- ... applied to two arguments
```

λ-calculus: Booleans

How can we encode Boolean values (`TRUE` and `FALSE`) as functions?

Well, what do we **do** with a Boolean `b` ?

Make a *binary choice*

- `if b then e1 else e2`

Booleans: API

We need to define three functions

```
let TRUE  = ???
let FALSE = ???
let ITE   = \b x y -> ???  -- if b then x else y
```

such that

```
ITE TRUE apple banana =~> apple
ITE FALSE apple banana =~> banana
```

(Here, `let NAME = e` means `NAME` is an *abbreviation* for `e`)

Booleans: Implementation

```
let TRUE  = \x y -> x        -- Returns its first argument
let FALSE = \x y -> y        -- Returns its second argument
let ITE   = \b x y -> b x y  -- Applies condition to branches
                             -- (redundant, but improves readability)
```

Example: Branches step-by-step

```
eval ite_true:
  ITE TRUE e1 e2
  =d> (\b x y -> b    x  y) TRUE e1 e2    -- expand def ITE
  =b>   (\x y -> TRUE x  y)      e1 e2    -- beta-step
  =b>     (\y -> TRUE e1 y)         e2    -- beta-step
  =b>            TRUE e1 e2               -- expand def TRUE
  =d>     (\x y -> x) e1 e2               -- beta-step
  =b>       (\y -> e1)   e2               -- beta-step
  =b> e1
```

Example: Branches step-by-step

Now you try it!

Can you fill in the blanks to make it happen?

```
eval ite_false:
  ITE FALSE e1 e2

  -- fill the steps in!

  =b> e2
```

EXERCISE: Boolean Operators

ELSA: https://goto.ucsd.edu/elsa/index.html Click here to try this exercise

Now that we have `ITE` it's easy to define other Boolean operators:

```
let NOT = \b     -> ???
let OR  = \b1 b2 -> ???
let AND = \b1 b2 -> ???
```

When you are done, you should get the following behavior:

```
eval ex_not_t:
  NOT TRUE =*> FALSE

eval ex_not_f:
  NOT FALSE =*> TRUE

eval ex_or_ff:
  OR FALSE FALSE =*> FALSE

eval ex_or_ft:
  OR FALSE TRUE =*> TRUE

eval ex_or_ft:
  OR TRUE FALSE =*> TRUE

eval ex_or_tt:
  OR TRUE TRUE =*> TRUE

eval ex_and_ff:
  AND FALSE FALSE =*> FALSE

eval ex_and_ft:
  AND FALSE TRUE =*> FALSE

eval ex_and_ft:
  AND TRUE FALSE =*> FALSE

eval ex_and_tt:
  AND TRUE TRUE =*> TRUE
```

Programming in λ-calculus

- **Booleans** [done]
- Records (structs, tuples)
- Numbers
- Lists
- **Functions** [we got those]
- Recursion

λ-calculus: Records

Let's start with records with *two* fields (aka **pairs**)

What do we *do* with a pair?

1. **Pack two** items into a pair, then
2. **Get first** item, or
3. **Get second** item.

Pairs : API

We need to define three functions

```
let PAIR = \x y -> ???    -- Make a pair with elements x and y
                          -- { fst : x, snd : y }
let FST  = \p -> ???      -- Return first element
                          -- p.fst
let SND  = \p -> ???      -- Return second element
                          -- p.snd
```

such that

```
eval ex_fst:
  FST (PAIR apple banana) =*> apple

eval ex_snd:
  SND (PAIR apple banana) =*> banana
```

Pairs: Implementation

A pair of `x` and `y` is just something that lets you pick between `x` and `y` !

```
let PAIR = \x y -> (\b -> ITE b x y)
```

i.e. `PAIR x y` is a function that

- takes a boolean and returns either `x` or `y`

We can now implement `FST` and `SND` by "calling" the pair with `TRUE` or `FALSE`

```
let FST  = \p -> p TRUE   -- call w/ TRUE, get first value
let SND  = \p -> p FALSE  -- call w/ FALSE, get second value
```

EXERCISE: Triples

How can we implement a record that contains **three** values?

ELSA: https://goto.ucsd.edu/elsa/index.html

Click here to try this exercise

```
let TRIPLE = \x y z -> ???
let FST3   = \t -> ???
let SND3   = \t -> ???
let THD3   = \t -> ???

eval ex1:
  FST3 (TRIPLE apple banana orange)
  =*> apple

eval ex2:
  SND3 (TRIPLE apple banana orange)
  =*> banana

eval ex3:
  THD3 (TRIPLE apple banana orange)
  =*> orange
```

Programming in λ-calculus

- **Booleans** [done]
- **Records** (structs, tuples) [done]
- Numbers
- Lists
- **Functions** [we got those]
- Recursion

"3" \f x → f (f (f x))
"1" \f x → f x
"2" \f x → f (f x)
"n" \f x → $\underbrace{f(\dots f}_{n}(x))$

λ-calculus: Numbers

Let's start with **natural numbers** (0, 1, 2, ...)

What do we *do* with natural numbers?

- Count: `0` , `inc`
- Arithmetic: `dec` , `+` , `-` , `*`
- Comparisons: `==` , `<=` , etc

Natural Numbers: API

We need to define:

- A family of **numerals**: `ZERO` , `ONE` , `TWO` , `THREE` , ...
- Arithmetic functions: `INC` , `DEC` , `ADD` , `SUB` , `MULT`
- Comparisons: `IS_ZERO` , `EQ`

Such that they respect all regular laws of arithmetic, e.g.

```
IS_ZERO ZERO       =~> TRUE
IS_ZERO (INC ZERO) =~> FALSE
INC ONE            =~> TWO
...
```

INC = \n → ??

\x1 x2 → x1
\x1 x2 → x2

Natural Numbers: Implementation

Church numerals: *a number* `N` is encoded as a combinator that *calls a function on an argument* `N` *times*

let ZERO = \f x → x

```
let ONE   = \f x -> f x
let TWO   = \f x -> f (f x)
let THREE = \f x -> f (f (f x))
let FOUR  = \f x -> f (f (f (f x)))
let FIVE  = \f x -> f (f (f (f (f x))))
let SIX   = \f x -> f (f (f (f (f (f x)))))
...
```

let N92 = \f x → $\underbrace{f \dots f}_{92}(f x)$

(n f x) ⇒ $\underbrace{f \dots f (f (f}_{n} x))$

QUIZ: Church Numerals

Which of these is a valid encoding of `ZERO` ?

- **A**: `let ZERO = \f x -> x`
- **B**: `let ZERO = \f x -> f`
- **C**: `let ZERO = \f x -> f x`
- **D**: `let ZERO = \x -> x`
- **E**: None of the above

Does this function look familiar?

(n f x) ≡ "call f on x n times"

λ-calculus: Increment

```
-- Call `f` on `x` one more time than `n` does
let INC   = \n -> (\f x -> ???)
```

INC N0 ⇒ N1
INC N2 ⇒ N3
INC N3 ⇒ N4

INC n = \f x → f (n f x)

Example:

```
eval inc_zero :
  INC ZERO
  =d> (\n f x -> f (n f x)) ZERO
  =b> \f x -> f (ZERO f x)
  =*> \f x -> f x
  =d> ONE
```

EXERCISE

Fill in the implementation of `ADD` so that you get the following behavior

Click here to try this exercise

```
let ZERO = \f x -> x
let ONE  = \f x -> f x
let TWO  = \f x -> f (f x)
let INC  = \n f x -> f (n f x)

let ADD  = fill_this_in

eval add_zero_zero:
  ADD ZERO ZERO =~> ZERO

eval add_zero_one:
  ADD ZERO ONE =~> ONE

eval add_zero_two:
  ADD ZERO TWO =~> TWO

eval add_one_zero:
  ADD ONE ZERO =~> ONE

eval add_one_zero:
  ADD ONE ONE =~> TWO

eval add_two_zero:
  ADD TWO ZERO =~> TWO
```

QUIZ

How shall we implement `ADD` ?

A. `let ADD = \n m -> n INC m`

B. `let ADD = \n m -> INC n m`

C. `let ADD = \n m -> n m INC`

D. `let ADD = \n m -> n (m INC)`

E. `let ADD = \n m -> n (INC m)`

λ-calculus: Addition

```
--  Call `f` on `x` exactly `n + m` times
let ADD = \n m -> n INC m
```

Example:

```
eval add_one_zero :
  ADD ONE ZERO
  =~> ONE
```

QUIZ

How shall we implement `MULT` ?

A. `let MULT = \n m -> n ADD m`

B. `let MULT = \n m -> n (ADD m) ZERO`

C. `let MULT = \n m -> m (ADD n) ZERO`

D. `let MULT = \n m -> n (ADD m ZERO)`

E. `let MULT = \n m -> (n ADD m) ZERO`

λ-calculus: Multiplication

```
--  Call `f` on `x` exactly `n * m` times
let MULT = \n m -> n (ADD m) ZERO
```

Example:

```
eval two_times_three :
  MULT TWO ONE
  =~> TWO
```

Programming in λ-calculus

- **Booleans** [done]
- **Records** (structs, tuples) [done]
- **Numbers** [done]
- $\boxed{\textbf{Lists}}$
- **Functions** [we got those]
- Recursion

λ-calculus: Lists

Lets define an API to build lists in the λ-calculus.

An Empty List

```
NIL
```

Constructing a list

A list with 4 elements

```
CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

intuitively `CONS h t` creates a *new* list with

- *head* `h`
- *tail* `t`

Destructing a list

- `HEAD l` returns the *first* element of the list
- `TAIL l` returns the *rest* of the list

```
HEAD (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
=~> apple

TAIL (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
=~> CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

λ-calculus: Lists

```
let NIL  = ???
let CONS = ???
let HEAD = ???
let TAIL = ???

eval exHd:
  HEAD (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
  =~> apple

eval exTl
  TAIL (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
  =~> CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

EXERCISE: Nth

Write an implementation of `GetNth` such that

- `GetNth n l` returns the n-th element of the list `l`

Assume that `l` has n or more elements

```
let GetNth = ???

eval nth1 :
  GetNth ZERO (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> apple

eval nth1 :
  GetNth ONE (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> banana

eval nth2 :
  GetNth TWO (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> cantaloupe
```

Click here to try this in elsa

def sum(n):
  res = 0, i = 0
  for n-times:
    (res, i) ← (res+i, i+1)
  return res

n (\p →) (PACK 0 0)

λ-calculus: Recursion

I want to write a function that sums up natural numbers up to `n` :

```
let SUM = \n -> ...   -- 0 + 1 + 2 + ... + n
```

such that we get the following behavior

```
eval exSum0: SUM ZERO  =~> ZERO
eval exSum1: SUM ONE   =~> ONE
eval exSum2: SUM TWO   =~> THREE
eval exSum3: SUM THREE =~> SIX
```

n (\p →
  PACK (ADD (FST p) (SND p)) (INC (SND p))
)
(PACK 0 0)

Can we write sum **using Church Numerals**?

Click here to try this in Elsa

QUIZ

You *can* write `SUM` using numerals but its *tedious*.

Is this a correct implementation of `SUM` ?

```
let SUM = \n -> ITE (ISZ n)
            ZERO
            (ADD n (SUM (DEC n)))
```

A. Yes

B. No

No!

- Named terms in Elsa are just syntactic sugar
- To translate an Elsa term to λ-calculus: replace each name with its definition

```
\n -> ITE (ISZ n)
         ZERO
         (ADD n (SUM (DEC n))) -- But SUM is not yet defined!
```

Recursion:

- Inside *this* function
- Want to call the *same* function on `DEC n`

Looks like we can't do recursion!

- Requires being able to refer to functions *by name*,
- But λ-calculus functions are *anonymous*.

Right?

λ-calculus: Recursion

Think again!

Recursion:

Instead of

- ~~Inside *this* function I want to call the *same* function on `DEC n`~~

Lets try

- Inside *this* function I want to call *some* function `rec` on `DEC n`
- And BTW, I want `rec` to be the *same* function

Step 1: Pass in the function to call "recursively"

```
let STEP =
  \rec -> \n -> ITE (ISZ n)
                  ZERO
                  (ADD n (rec (DEC n))) -- Call some rec
```

Step 2: Do some magic to `STEP`, so `rec` is itself

```
\n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))
```

That is, obtain a term `MAGIC` such that

```
MAGIC =*> STEP MAGIC
```

λ-calculus: Fixpoint Combinator

Wanted: a λ-term `FIX` such that

- `FIX STEP` calls `STEP` with `FIX STEP` as the first argument:

```
(FIX STEP) =*> STEP (FIX STEP)
```

(In math: a *fixpoint* of a function $f(x)$ is a point x, such that $f(x) = x$)

Once we have it, we can define:

```
let SUM = FIX STEP
```

Then by property of `FIX` we have:

```
SUM   =*>   FIX STEP  =*>   STEP (FIX STEP)   =*>   STEP SUM
```

and so now we compute:

```
eval sum_two:
  SUM TWO
  =*> STEP SUM TWO
  =*> ITE (ISZ TWO) ZERO (ADD TWO (SUM (DEC TWO)))
  =*> ADD TWO (SUM (DEC TWO))
  =*> ADD TWO (SUM ONE)
  =*> ADD TWO (STEP SUM ONE)
  =*> ADD TWO (ITE (ISZ ONE) ZERO (ADD ONE (SUM (DEC ONE))))
  =*> ADD TWO (ADD ONE (SUM (DEC ONE)))
  =*> ADD TWO (ADD ONE (SUM ZERO))
  =*> ADD TWO (ADD ONE (ITE (ISZ ZERO) ZERO (ADD ZERO (SUM DEC ZERO)))
  =*> ADD TWO (ADD ONE (ZERO))
  =*> THREE
```

How should we define `FIX`???

The Y combinator

Remember Ω?

```
(\x -> x x) (\x -> x x)
=b> (\x -> x x) (\x -> x x)
```

This is *self-replcating code*! We need something like this but a bit more involved…

The Y combinator discovered by Haskell Curry:

```
let FIX   = \stp -> (\x -> stp (x x)) (\x -> stp (x x))
```

How does it work?

```
eval fix_step:
  FIX STEP
  =d> (\stp -> (\x -> stp (x x)) (\x -> stp (x x))) STEP
  =b> (\x -> STEP (x x)) (\x -> STEP (x x))
  =b> STEP ((\x -> STEP (x x)) (\x -> STEP (x x)))
  --        ^^^^^^^^^^ this is FIX STEP ^^^^^^^^^^^
```

That's all folks, Haskell Curry was very clever.

Next week: We'll look at the language named after him (`Haskell`)